Ground state overlap and quantum phase transitions

Paolo Zanardi and Nikola Paunković*
Institute for Scientific Interchange (ISI), Villa Gualino, Viale Settimio Severo 65, I-10133 Torino, Italy
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We present a characterization of quantum phase transitions in terms of the the overlap function between two ground states obtained for two different values of external parameters. On the examples of the Dicke and *XY* models, we show that the regions of criticality of a system are marked by the extremal points of the overlap and functions closely related to it. Further, we discuss the connections between this approach and the Anderson orthogonality catastrophe as well as with the dynamical study of the Loschmidt echo for critical systems.

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INTRODUCTION

Quantum phase transitions (QPTs) [1] have drawn a considerable interest within various fields of physics in the recent years. They are studied in condensed matter physics because they provide valuable information about the novel type of finite-temperature states of matter that emerge in the vicinity of QPTs [1]. Unlike the ordinary phase transitions, driven by thermal fluctuations, QPTs occur at zero temperature and are driven by purely quantum fluctuations. In the parameter space, the points of nonanalyticity of the ground state energy density are referred to as critical points and define the QPTs. In these points one typically witnesses the divergence of the length associated to the two-point correlation function of some relevant quantum field. An alternative way of characterizing QPTs is by the vanishing, in the thermodynamical limit, of the energy gap between the ground and the first excited state in the critical points. Recently, a huge interest was raised in the attempt of characterizing QPTs in terms of the notions and tools of quantum information [2]. More specifically QPTs have been studied by analyzing scaling, asymptotic behavior, and extremal points of various entanglement measures [3–7]. More recently, the connection between geometric Berry phases and QPTs in the case of the *XY* model has been also studied [8].

In this paper, we aim to provide yet another characterization of the regions of criticality that define QPTs. We shall show how critical points can be individuated by studying a surprisingly simple quantity: the overlap, i.e., the scalar product, between two ground states corresponding to two slightly different values of the parameters. The physical intuition behind this approach should be obvious: QPTs mark the separation between regions of the parameter space which correspond to ground states having deeply different structural properties, e.g., order parameters. This difference is here quantified by the simplest Hilbert-space geometrical quantity, i.e., the overlap. Note that the square modulus of the overlap is nothing but the fidelity, widely used in quantum information as a function that provides the criterion for distinguishability between quantum states [2]. Therefore it is a natural candidate for a study of macroscopic distinguishability between quantum states that define different macroscopic states of matter (different phases). When applied to cases of many-body systems containing many degrees of freedom, the overlap (or, fidelity) might seem to be too coarse a quantity, and not bearing any apparent information about the difference in order properties between quantum phases to be of any use. Nevertheless, the main result of this paper is that in some cases it is indeed possible to do so. The critical behavior of a system undergoing QPTs is reflected in the geometry of its Hilbert space: approaching the QPTs the overlap (distance) between neighboring ground states shows a dramatic drop (increase). We would like to notice that Cejnar *et al.* [9] discussed the overlap entropy between the eigenbases of a system's Hamiltonian and various physically relevant bases, in the context of enhanced decoherence effects in the regions of criticality (see also Ref. [10]).

In the following two sections, we conduct our analysis on the cases of two simple, yet physically relevant and mathematically instructive, examples of the Dicke model and the *XY* spin-chain model. Next, we discuss the connection between the scaling and asymptotic behaviors and the so-called Anderson orthogonality catastrophe [11]. Moreover, the relation with the dynamical study of decoherence and quantum criticality [12] is briefly addressed. Finally, in the last section conclusions are discussed.

For a generic point in parameter space we use the label $q \in \mathbb{R}^L$, where L is the number of external parameters determining the system's Hamiltonian. As the overlap function depends on the difference between parameters as well, we introduce $\tilde{q} \equiv q + \delta q$ to denote the neighboring point $\tilde{q}$ and the difference δq. Following this notation, we denote the ground states by $|g\rangle \equiv |g(q)\rangle$ and $|\tilde{q}\rangle \equiv |g(\tilde{q})\rangle$. In general, all the functions $F(q)$ evaluated in the point $\tilde{q}$ we will denote as $\tilde{F}$, while those evaluated in the critical point q_c we will denote as F_c [note that by combining two cases, we have $\tilde{F}_c = F(q_c + \delta q)$]. Then, the overlap function is simply given by the scalar product $\langle g(q) | g(\tilde{q}) \rangle$ (note that all the results of this paper could be easily formulated in terms of fidelity as well). We shall examine the behavior of the overlap as a function of q only, while keeping δq fixed and small.

*Present address: SQIG, Instituto de Telecomunicações and Instituto Superior Técnico, P-1049-001, Lisboa, Portugal.

DICKE MODEL

Our first example is the Dicke model. It describes a dipole interaction between a single Bosonic mode $\hat{a}$ and a collection of N two-level atoms. If for N atoms we introduce the collective angular momentum operators $\hat{J}_s, s \in \{\pm, z\}$, the Dicke Hamiltonian has the following form (we take $\hbar = 1$):

$$\hat{H}(\lambda) = \omega_0 \hat{J}_z + \omega \hat{a}^\dagger \hat{a} + \frac{\lambda}{\sqrt{2j}}(\hat{a}^\dagger + \hat{a})(\hat{J}_+ + \hat{J}_-). \tag{1}$$

The parameter λ is the atom-field coupling strength and is the one driving the QPTs in this model. Therefore we have $q=\lambda$ and denote the Hamiltonian's dependence on that parameter only. Parameters ω_0 and ω stand for the atomic level-splitting and Bosonic mode frequency, respectively, while j describes the length of a collective spin vector, and is assumed to be constant and equal to $j=N/2$. In the thermodynamical limit ($N \to \infty$), which is here equivalent to ($j \to \infty$), the Dicke Hamiltonian undergoes a quantum phase transition for the critical value of its parameter λ given by $\lambda_c = \sqrt{\omega\omega_0}/2$. When $\lambda < \lambda_c$, the system is in a highly unexcited *normal* phase, while $\lambda > \lambda_c$ defines the *super-radiant* phase in which both the field and N atoms become macroscopically excited. The super-radiant phase is characterized by the broken symmetry given by the parity operator $\hat{\Pi} = \exp(i\pi\hat{N})$, $\hat{N} = (\hat{a}^\dagger\hat{a} + \hat{J}_z + j)$: the ground state is doubly degenerate. As shown in Ref. [13], by introducing Bosonic operators $\hat{b}$ through the Holstein-Primakoff representation [14], the above Dicke Hamiltonian (1) can be exactly diagonalized in the thermodynamical limit. In the normal phase, its form is

$$\hat{H}^n(\lambda) = \omega_0 \hat{b}^\dagger \hat{b} + \omega \hat{a}^\dagger \hat{a} + \lambda(\hat{a}^\dagger + \hat{a})(\hat{b}^\dagger + \hat{b}) - j\omega_0. \tag{2}$$

Its ground state is $g(x,y) = \left(\frac{\varepsilon_+\varepsilon_-}{\pi^2}\right)^{1/4} e^{-1/2\langle \mathbf{R}, A\mathbf{R}\rangle}[\mathbf{R}=(x,y)]$ where x and y are the real-space coordinates associated to the modes $\hat{a}$ and $\hat{b}$, $A = U^{-1}MU$, $M = \mathrm{diag}[\varepsilon_-, \varepsilon_+]$ and U an orthogonal matrix

$$U = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

[$c = \cos\gamma, s = \sin\gamma$ are given by the squeezing angle $\gamma = (1/2)\arctan[4\lambda\sqrt{\omega\omega_0}/(\omega^2+\omega_0^2)]$]. $\varepsilon_\pm$ represent the fundamental collective excitations of the system and are given by $\varepsilon_\pm^2 = \frac{1}{2}(\omega^2 + \omega_0^2 \pm \sqrt{(\omega^2-\omega_0^2)^2 + 16\lambda^2\omega\omega_0})$. From the above formula, we see that $\varepsilon_-(\lambda_c) \equiv \varepsilon_-^c = 0$: the system becomes gapless and undergoes a QPT for $\lambda = \lambda_c$.

The overlap, calculated between two ground states g and $\tilde{g}$, is given by

$$\langle g|\tilde{g}\rangle = 2\frac{[\det A \det \tilde{A}]^{1/4}}{[\det(A+\tilde{A})]^{1/2}} = 2\frac{[\det A]^{1/4}}{[\det \tilde{A}]^{1/4}[\det(1+\tilde{A}^{-1}A)]^{1/2}}. \tag{3}$$

Note that the overlap is a function of both λ and $\delta\lambda$. In the limit ($\lambda \to \lambda_c$), with $\delta\lambda > 0$ being fixed, $\det A = \varepsilon_+\varepsilon_- \to 0$, while $\det \tilde{A} \geq \det \tilde{A}_c = \tilde{\varepsilon}_+^c \tilde{\varepsilon}_-^c > 0$. The same holds for $\det(1+\tilde{A}^{-1}A)$, for a sufficiently small $\delta\lambda$ (note that $\lim_{\delta\lambda\to 0}\tilde{A}^{-1} = A^{-1}$). But in the present case, it is possible to show that for every, and not just small, $\delta\lambda$, $\det(1+\tilde{A}^{-1}A)$ does not vanish. Using the formula $\det(1+A) = 1 + \mathrm{Tr}A + \det A$ for 2×2 matrices, we get $\det(1+\tilde{A}^{-1}A) \to 1 + \mathrm{Tr}(\tilde{A}_c^{-1}A_c)$ [note that $\det(\tilde{A}^{-1}A) = [\det\tilde{A}]^{-1}\det A \to 0$]. After a straightforward calculation, we obtain the result: $\mathrm{Tr}(\tilde{A}_c^{-1}A_c) = \frac{\varepsilon_+^c}{\tilde{\varepsilon}_+^c\tilde{\varepsilon}_-^c}[(s\tilde{c}+c\tilde{s})^2\tilde{\varepsilon}_+^c + (s\tilde{s}-c\tilde{c})^2\tilde{\varepsilon}_-^c]$. Therefore $\mathrm{Tr}(\tilde{A}_c^{-1}A_c) > 0$ for every λ and we can conclude that $\langle g|\tilde{g}\rangle \propto (\varepsilon_-)^{1/4}$ as ($\lambda \to \lambda_c$). In Ref. [13], it was shown that when approaching the critical point from both normal and super-radiant sides, the excitation energy ε_- drops as the square root of $\Delta \equiv |\lambda - \lambda_c|$, which gives us the asymptotic behavior of the overlap function in the vicinity of the critical point: $\langle g|\tilde{g}\rangle \propto \Delta^{1/8}$. Although we have provided here only the results for overlap function for the system in the normal phase, the analogous analysis for the super-radiant phase gives us the same qualitative results, as the two ground states are again the Gaussian-type states, but with translated and re-scaled x and y axes. Therefore we omit it here.

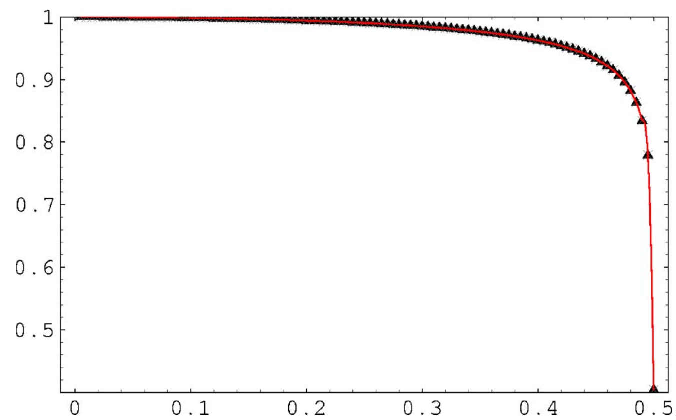

FIG. 1. (Color online) The overlap function $\langle g|\tilde{g}\rangle$, Eq. (3), as a function of $\lambda < \lambda_c$, taken for the resonant case $\omega_0 = \omega = 1$ and $\delta\lambda = 10^{-6}$. Note the dramatic decreasing of the function as we approach the point of criticality.

We conclude this section by presenting the numerical results for the overlap function in the normal phase. In Fig. 1 we plot the overlap (3) between two ground states of the Dicke Hamiltonian for the resonant case $\omega_0 = \omega = 1$ and $\delta\lambda = 10^{-6}$. We see that it is almost constant and equal to unity for a wide range of λ, apart from the very narrow area around λ_c, when it drastically drops to zero. Such behavior of the overlap function around the point of criticality can be ascribed to the fact that the ground state for $\lambda = \lambda_c$ becomes completely delocalized along one of two rotated axes, as opposed to the localized ground state outside of the point of criticality (see Ref. [13]).

XY SPIN CHAIN

In the following section, we discuss the example of the one-dimensional *XY* anisotropic spin-half chain in the external magnetic field. Its Hamiltonian is given by the following expression:

$$\hat{H}(\gamma,\lambda) = -\sum_{i=-M}^{M}\left(\frac{1+\gamma}{2}\hat{\sigma}_i^x\hat{\sigma}_{i+1}^x + \frac{1-\gamma}{2}\hat{\sigma}_i^y\hat{\sigma}_{i+1}^y + \frac{\lambda}{2}\hat{\sigma}_i^z\right). \tag{4}$$

The parameter $\gamma \in \mathbb{R}$ defines the anisotropy, while $\lambda \in \mathbb{R}$ represents external magnetic field along the z axis, up to a factor $\frac{1}{2}$. Therefore $q=(\gamma,\lambda)$. The operators $\hat{\sigma}_i^\alpha$, $\alpha \in \{x,y,z\}$ are the usual Pauli operators. This Hamiltonian can be exactly diagonalized by successively applying Jordan-Wigner, Furier, and Bogoliubov transformation (see, for example, Ref. [1]). This way, we obtain the following form of the Hamiltonian: $\hat{H}(\gamma,\lambda)=\sum_{k=-M}^{M}\Lambda_k(\hat{b}_k^\dagger\hat{b}_k-1)$. The energies of one-particle excitations are given by $\Lambda_k=\sqrt{\varepsilon_k^2+\gamma^2\sin^2\frac{2\pi k}{N}}$, with $\varepsilon_k=\cos\frac{2\pi k}{N}-\lambda$ and $N=2M+1$ being the total number of sites (spins). One-particle excitations are given by the Fermionic operators $\hat{b}_k=\cos\frac{\theta_k}{2}\hat{d}_k-i\sin\frac{\theta_k}{2}\hat{d}_{-k}^\dagger$, with $\cos\theta_k=\varepsilon_k/\Lambda_k$. Finally, the ground state $|g(\gamma,\lambda)\rangle$, that is defined as the state to be annihilated by each operator $\hat{b}_k$ $[\hat{b}_k|g(\gamma,\lambda)\rangle\equiv 0]$, is given as a tensor product of qubitlike states:

$$|g(\gamma,\lambda)\rangle = \bigotimes_{k=1}^{M}\left(\cos\frac{\theta_k}{2}|0\rangle_k|0\rangle_{-k} - i\sin\frac{\theta_k}{2}|1\rangle_k|1\rangle_{-k}\right). \tag{5}$$

In its space of parameters, the family of Hamiltonians given by Eq. (4) exhibits two regions of criticality, defined by the existence of gapless excitations: (i) the *XX* region of criticality, for $\gamma=0$ and $\lambda\in(-1,1)$; (ii) the *XY* region of criticality given by the lines $\lambda=\pm 1$.

As in the previous example, let us first consider the exact overlap function. From Eq. (5), it follows that the exact overlap function between the ground states $|g\rangle$ and $|\tilde{q}\rangle$ is

$$\langle g(q)|g(\tilde{q})\rangle = \prod_{k=1}^{M}\cos\frac{\theta_k-\tilde{\theta}_k}{2}, \tag{6}$$

where $\tilde{\theta}_k=\theta_k(\tilde{q})$. Note the dependence on the number of sites N that is implicit in all the previous formulas from this section. In Fig. 2(a), we present the numerical result obtained using the above Eq. (6), for $N=10^6$ spins and $\delta\lambda=\delta\gamma=10^{-6}$. We observe that the regions of criticality are clearly marked by a sudden drop of the value of the overlap function. As before, we ascribe this type of behavior to a dramatic change in the structure of the ground state of the system while undergoing QPTs.

In order to investigate the overlap function more quantitatively and relate its behavior to the existence of the regions of criticality, we note that while the overlap depends on the values of both the parameters q and the difference δq, the regions of criticality are defined by the values of parameters only. Therefore in the following we choose to study the functions

$$S_N^\lambda(\lambda,\gamma) \equiv \sum_{k=1}^{M}\left(\frac{\partial\theta_k}{\partial\lambda}\right)^2, \quad S_N^\gamma(\lambda,\gamma) \equiv \sum_{k=1}^{M}\left(\frac{\partial\theta_k}{\partial\gamma}\right)^2, \tag{7}$$

that define the first nonzero order of the Taylor expansion of the overlap function (6). The functions $S_N^\lambda(\lambda,\gamma)$ and $S_N^\gamma(\lambda,\gamma)$ are natural candidates for our study because they express the "rate of change" of the ground state, taken in the point q. They do not depend on the difference δq, and although for every finite N it is possible to find δq small enough so that the exact overlap is arbitrarily well approximated by the expression $\exp[-\frac{1}{8}S_N^q(q)\delta q^2]$, the functions $S_N^\lambda(\lambda,\gamma)$ and $S_N^\gamma(\lambda,\gamma)$ on their own capture the behavior of the $\langle g(q)|g(\tilde{q})\rangle$ function and are enough for our current study. They also allow for analytic investigation, together with numerical one. In Figs. 2(b) and 2(c) we present the numerical results for $S_N^\lambda(\lambda,\gamma)$ and $S_N^\gamma(\lambda,\gamma)$, respectively, for $N=10^6$ spins. Again, the regions of criticality could easily be inferred by simply observing both plots. Note that in this case the relative difference between the numerical values in the regions of criticality and elsewhere is much bigger than in the case of the exact overlap [see Fig. 2(a)]. But now, *both* plots are needed to detect both regions of criticality. This is so because by moving along $\gamma=0$, while keeping $|\lambda|<1$, we do not move outside the *XX* region of criticality and therefore do not expect the qualitative change in the structure of the ground state, and consequently in the behavior of $S_N^\lambda(\lambda,\gamma)$ as well. The same holds for $S_N^\gamma(\lambda,\gamma)$ and the *XY* region of criticality.

We first examine the scaling behavior of $S_N^\lambda(\lambda,\gamma)$ and $S_N^\gamma(\lambda,\gamma)$ with respect to number of spins N. The numerics present us with the following results. First, as expected, $S_N^\lambda(\lambda,\gamma)$ and $S_N^\gamma(\lambda,\gamma)$ scale linearly with N when $(\gamma\to 0)$ and $(\lambda\to\pm 1)$, respectively. In the regions of criticality, we have that $S_N^\lambda(|\lambda|=1,\gamma)\propto N^2/\gamma^2$ and $S_N^\gamma(|\lambda|\leqslant 1,\gamma=0)\propto N^2$ [we note here that $S_N^\lambda(|\lambda|\leqslant 1,\gamma=0)=0$].

We have also conducted a separate analytical study, confirming the above numerical results. First, we note that for every point q in parameter space, and every *finite* N, partial derivatives $\left(\frac{\partial\theta_k}{\partial\lambda}\right)$ and $\left(\frac{\partial\theta_k}{\partial\gamma}\right)$ are continuous functions of the parameters. They can become infinite only in the thermodynamical limit, when $(N\to\infty)$, and only in the *regions of criticality*. By looking at the explicit form of derivatives (we use $x_k=\frac{2\pi}{N}k$):

$$\left(\frac{\partial\theta_k}{\partial\lambda}\right) = \frac{\gamma(\sin x_k)}{[(\cos x_k-\lambda)^2+\gamma^2(\sin x_k)^2]}, \tag{8}$$

$$\left(\frac{\partial\theta_k}{\partial\gamma}\right) = -\frac{|\sin x_k|(\cos x_k-\lambda)}{[(\cos x_k-\lambda)^2+\gamma^2(\sin x_k)^2]}, \tag{9}$$

we see that only when the energy Λ_k (the denominator of both of the above expressions) gets arbitrarily small (or zero), the derivatives (8) and (9) can become divergent. In other words, only when $\cos x_k$ gets arbitrarily close to λ, *and* either γ or $\sin x_k$ get close to zero. That is, in the regions of criticality. Note that we assume that for every $N\in\mathbb{N}$ and $k\in\{1,\ldots M\}$, equation $\cos x_k=\lambda$ has no solution, which presents a generic case [λ's that allow for the solutions of this

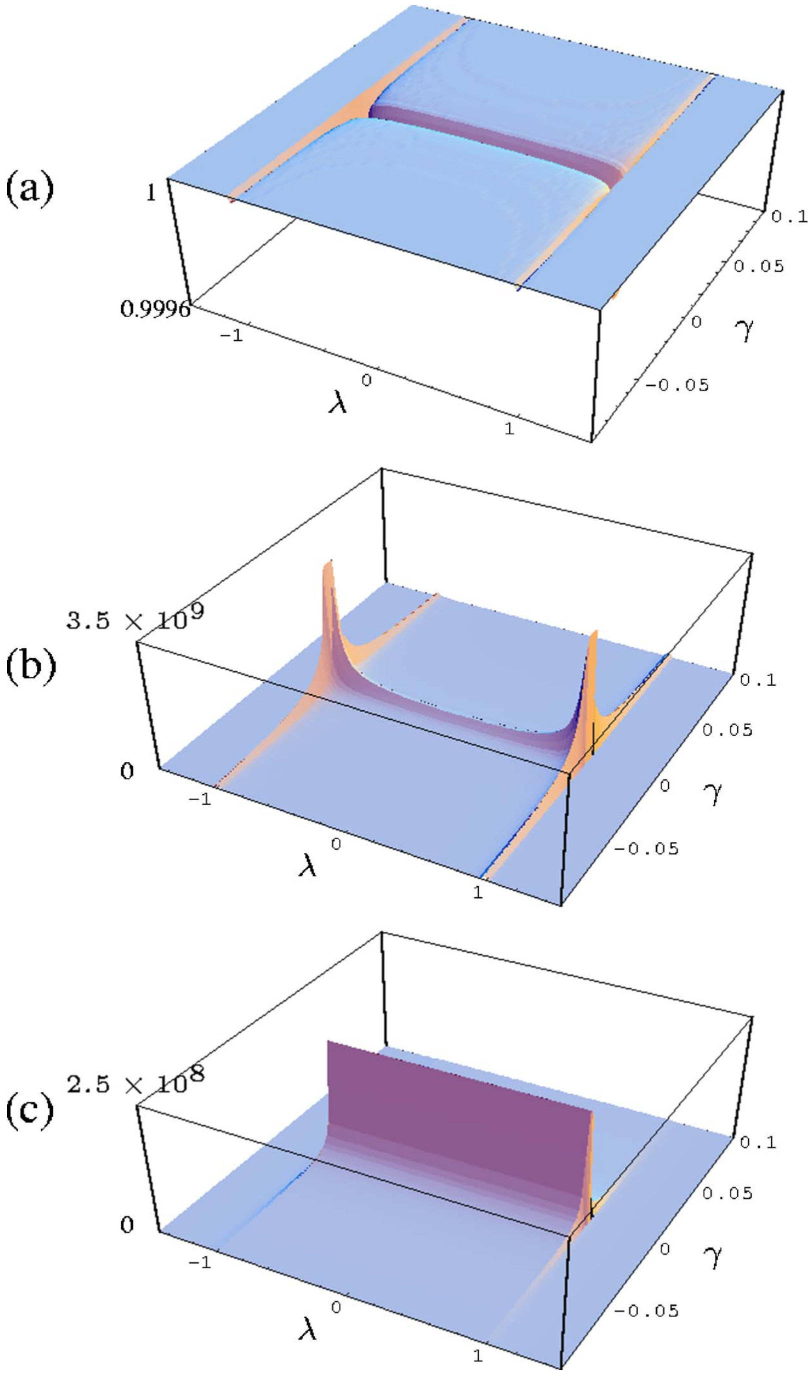

FIG. 2. (Color online) (a) The overlap function $\langle g(q)|g(\tilde{q})\rangle$, as a function of λ and γ, for $N=10^6$ and $\delta\lambda=\delta\gamma=10^{-6}$. Note the clear dips of the plot in the regions of criticality. (b) $S_N^\lambda(\lambda,\gamma)$. (c) $S_N^\gamma(\lambda,\gamma)$.

equation form a set of measure zero on the $(-1,1)$ interval]. Therefore outside the regions of criticality $S_N^\lambda(\lambda,\gamma)$ and $S_N^\gamma(\lambda,\gamma)$ scale linearly with N.

Regarding the regions of criticality, we first consider the scaling behavior of $S_N^\lambda(\lambda,\gamma)$ in the vicinity of the XX criticality. As there always exists k_0 such that in the $(N\to\infty)$ limit $\cos x_{k_0}\to\lambda$, then for such x_{k_0} and every finite γ, it follows from Eq. (8) that $\left(\frac{\partial\theta_{k_0}}{\partial\lambda}\right)\to(\gamma\sin x_{k_0})^{-1}$, when $(N\to\infty)$. In other words, it does not scale with N [note that although $k_0=k_0(N)$ is a function of N, $\lim_{N\to\infty}\sin x_{k_0}=\sin\arccos\lambda$]. As all other derivatives are finite, we have that $S_N^\lambda(|\lambda|<1,\gamma\to0)\propto N/\gamma^2$. Similar discussion can be applied to the case of $S_N^\gamma(\lambda,\gamma)$ in the XY region of criticality. Again, there exists a qubit defined by $k_1=1$ for which $\cos x_{k_1}\to1$ in the $(N\to\infty)$ limit, so that its existence could bring about the scaling of $S_N^\gamma(\lambda,\gamma)$ larger than linear in thermodynamical limit. Using the Taylor expansion of sine and cosine functions around zero (note that in that case, $\sin x_{k_1}\to0$ as well), from Eq. (9) we obtain $\left(\frac{\partial\theta_k}{\partial\gamma}\right)\propto x_{k_1}/\gamma^2\to0$, $(N\to\infty)$. In other words, $S_N^\gamma(|\lambda|=1,\gamma)\propto N$.

Now, we turn to more interesting cases of the relevant functions $S_N^\lambda(\lambda,\gamma)$ and $S_N^\gamma(\lambda,\gamma)$, in the XY and XX regions of criticality, respectively. Using Taylor expansions of sine and cosine functions around zero, we see that in $\lambda=\pm1$ the derivative $\left(\frac{\partial\theta_{k_1}}{\partial\lambda}\right)$ given by $k_1=1$ behaves like $\left(\frac{\partial\theta_{k_1}}{\partial\lambda}\right)\propto N/(2\pi\gamma)$ as $(N\to\infty)$ [see Eq. (8)] and therefore $S_N^\lambda(|\lambda|=1,\gamma)\propto N^2/\gamma^2$. We also see that the scaling factor depends on γ. Finally, from Eq. (9), we also see that $S_N^{\gamma}(|\lambda|\leqslant 1,\gamma=0)\propto N^2$.

The alternative way to examine the signatures of QPT is to look at the asymptotic behavior of two functions (7) near the regions of criticality. From the numerical study we obtain that the asymptotic behavior of $S_N^{\lambda}(\lambda,\gamma)$ in the vicinity of critical points $\lambda_c=\pm 1$, $\gamma\in(0,1]$, is given by the following formula: $S_N^{\lambda}(\lambda,\gamma)\propto a(\gamma,N)/|1-\lambda|^{\alpha(\gamma,N)}$. From the study of the scaling behavior, we already know that $a(\gamma,N)=a(\gamma)N^2$ and that $a(\gamma)\propto 1/\gamma^2$. Further, from numerics we have that the exponent $\alpha(\gamma,N)$ is constant with respect to γ and approaches to $\alpha=1$ as $(N\to\infty)$. Such asymptotic behavior, with constant exponent $\alpha=1$ for all $\gamma\in(0,1]$ could be seen as a consequence of the fact that in that range of parameters the *XY* model belongs to the same class of universality. The numerics gives also the asymptotic behavior of $S_N^{\gamma}(\lambda,\gamma)$ in the vicinity of $\gamma=0$ (with $|\lambda|<1$) similar to the previous one, $S_N^{\gamma}(\lambda,\gamma)\propto b(\lambda,N)/\gamma^{\beta(\lambda,N)}$, with the exponent $\beta(\lambda,N)$ approaching to $\beta=1$ as $(N\to\infty)$. But, the coefficient $b(\lambda,N)$ depends only on N, and as noted before, scales linearly with it, $b(\lambda,N)\propto bN$.

QPT: ORTHOGONALITY CATASTROPHE, LOSCHMIDT ECHO

The above two examples represent a generic case of a many-body system which exhibits continuous QPTs only in the thermodynamical limit. In the case of the *XY* model, as the number of spins increases, the overlap between two different ground states (5) approaches zero, no matter how small the difference in parameters δq is, so that in thermodynamical limit each two ground states are mutually orthogonal; they live in a continuous tensor product space [15]. Such behavior of systems having infinitely many degrees of freedom, when the two physical states corresponding to two arbitrarily close sets of parameters (two arbitrarily similar physical situations) become orthogonal to each other, has been already studied in many-body physics and is known as Anderson *orthogonality catastrophe* [11]. From our study of the *XY* model, we have seen not only that every two ground states become orthogonal in thermodynamical limit, but also the rate of "orthogonalization" between two ground states of large, but finite system, changes qualitatively and grows faster in the vicinity of the regions of criticality. This way, the regions of criticality of an infinite system are already marked by the scaling and asymptotic behavior of the relevant functions of a finite-size system. Loosely speaking, the regions of criticality of QPTs are given as regions where the orthogonality catastrophe is expressed on a qualitatively greater scale. Notice that recently, the occurrence of a particular instance of Anderson-type orthogonality catastrophe was studied for the case of a system in the vicinity of QPTs [16].

Now we would like to establish an explicit connection between the sort of kinematical approach used in this paper and the dynamical one of Ref. [12]. In order to do so let us introduce the projected density of states function $D(\omega;q,\tilde{q})\equiv\langle g(\tilde{q})|\delta[\omega-\hat{H}(q)]|g(\tilde{q})\rangle$ that describes the spread of the ground state $|g(\tilde{q})\rangle$ expressed in the eigenbasis obtained for the point q. Then, the square of the overlap can be expressed as

$$|\langle g(q)|g(\tilde{q})\rangle|^2 = 1-\int_{E_1}^{\infty} D(\omega)d\omega \tag{10}$$

(E_1 denotes the first excited eigenvalue). We see that in regions in which the spread of $|g(\tilde{q})\rangle$ with respect to $|g(q)\rangle$ is big [quantified in terms of the variance of $D(\omega)$], in other words, in regions where two ground states differ significantly, the overlap will be small. Recently, Quan *et al.* [12] established a link between the critical behavior of the environment and quantum decoherence, showing that the Loschmidt echo [17,18] $L(q,t)$ of the environment exponentially goes to zero as we approach the regions of criticality. A simple algebra gives us that the Fourier transform of the projected density of states is precisely the Loschmidt echo, $|\int_{-\infty}^{+\infty}D(\omega)e^{-i\omega t}d\omega|^2=L(q,t)$. We see that the kinematics of a system, given by the geometry of ground states through the overlap function, influences its dynamics as well: the smaller the overlap, i.e., broader $D(\omega)$, the faster the decay of the Loschmidt echo.

CONCLUSIONS AND DISCUSSION

In this paper, by discussing the examples of the Dicke and *XY* spin-chain models, we have presented a characterization of quantum phase transitions based on the study of the scaling and asymptotic behaviors of the overlap between two ground states taken in two close points of the parameter space. Though this quantity might in general not provide an efficient numerical tool it is conceptually quite appealing. In fact the ground state overlap is a purely Hilbert-space geometrical quantity, whose investigation does not rely on any *a priori* understanding of the specific kind of order patterns or peculiar dynamical correlations hidden in the analyzed system. While in the case of the Dicke Hamiltonian it was possible to analyze the overlap function directly in the thermodynamical limit, the case of the *XY* model is more subtle. The process of "orthogonalization" between two ground states has two physically different mechanisms in the case when two states belong to two different phases: one is a common decrease of the overlap due to infinite number of subsystems in thermodynamical limit, the other is characteristic for the case of QPTs and is due to different structures of ground states in different quantum phases.

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[1] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999).
[2] For reviews, see A. Steane, Rep. Prog. Phys. **61**, 117 (1998); D. P. DiVincenzo and C. H. Bennett, Nature (London) **404**, 247 (2000).
[3] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature (London) **416**, 608 (2002).
[4] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003).
[5] Y. Chen, P. Zanardi, Z. D. Wang, and F. C. Zhang, New J. Phys. **8**, 97 (2006).
[6] S.-J Gu, G.-S. Tian, and H.-Q. Lin, quant-ph/0509070.
[7] L.-A. Wu, M. S. Sarandy, and D. A. Lidar, Phys. Rev. Lett. **93**, 250404 (2004).
[8] A. C. M. Carollo and J. K. Pachos, Phys. Rev. Lett. **95**, 157203 (2005); Shi-Liang Zhu, *ibid.* **96**, 077206 (2006); A. Hamma, quant-ph/0602091.
[9] P. Cejnar, V. Zelevinsky, and V. V. Sokolov, Phys. Rev. E **63**, 036127 (2001).
[10] A. Volya and V. Zelevinsky, Phys. Lett. B **574**, 27 (2003); V. Zelevinsky, B. A. Brown, N. Fraizer, and M. Horoi, Phys. Rep. **276**, 85 (1996); P. Cejnar and J. Jolie, Phys. Rev. E **58**, 387 (1998); **61**, 6237 (2000).
[11] P. W. Anderson, Phys. Rev. Lett. **18**, 1049 (1967).
[12] H. T. Quan, Z. Song, X. F. Liu, P. Zanardi, and C. P. Sun, Phys. Rev. Lett. **96**, 140604 (2006).
[13] C. Emary and T. Brandes, Phys. Rev. E **67**, 066203 (2003).
[14] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1949).
[15] J. von Neumann, Comput. Appl. Math. **6**, 1 (1938); T. Thiemann and O. Winkler, Class. Quantum Grav. **18**, 4997 (2001).
[16] S. Sachdev, M. Troyer, and M. Vojta, Phys. Rev. Lett. **86**, 2617 (2001).
[17] P. R. Levstein, G. Usaj, and H. M. Pastawski, J. Chem. Phys. **108**, 2718 (1998); G. Usaj, H. M. Pastawski, and P. R. Levstein, Mol. Phys. **95**, 1229 (1998); H. M. Pastawski, P. R. Levstein, G. Usaj, J. Raya, and J. Hirschinger, Physica A **283**, 166 (2000).
[18] Z. P. Karkuszewski, C. Jarzynski, and W. H. Zurek, Phys. Rev. Lett. **89**, 170405 (2002); F. M. Cucchietti, D. A. R. Dalvit, J. P. Paz, and W. H. Zurek, *ibid.* **91**, 210403 (2003).